Dr. Cleary's Hopefully Helpful Guide to Symmetry and Integrals

Often if we have a region with some symmetry and a function with some symmetry we can save ourselves some work. Here are some details:

A **reflection** in Euclidean space (of any dimension) is transformation of Euclidean space which flips the points of space across some line or plane.

A typical example of a reflection is the transformation $x \to -x$. In the plane, this is reflection across the y-axis. In space, this is reflection across the yz-plane. That is, a point (x, y, z) gets mapped to the point $(-x, y, z)$ which is the mirror image of the point across the yz-plane. Other reflections are $y \to -y$ which is reflection across the xz plane and $z \to -z$ which is reflection across the xy-plane. For the xy plane, $y \to -y$ is reflection across the x-axis.

We say a region R is **symmetric with respect to the reflection** f if the image of R under the reflection is the same as R itself. This applies to R being a planar region, a solid, a surface, or any other kind of region. We also say R is **symmetric across a plane** if R is symmetric with respect to reflection across the plane.

For example, the unit disc D $\{x^2 + y^2 \leq 1\}$ is symmetric with respect to the reflection across the y-axis since the set $\{(-x)^2 + y^2 \leq 1\}$ is exactly the same set as D originally. That is, the set of points D is exactly the same set of points as $f(U)$.

In space, the unit sphere S $\{x^2 + y^2 + z^2 = 1\}$ is symmetric with respect to the reflection across the xy plane since the set $\{x^2 + y^2 + (-z)^2 = 1\}$ is exactly the same set as S originally. The unit sphere is also symmetric with respect to reflections across the xz and yz planes.
Here are more examples of regions in space:
Let B be the ball $x^2 + y^2 + z^2 \leq 1$. B is symmetric with respect to reflection across the xy, yz, and xz planes.

Let H be the solid hemisphere $x^2 + y^2 + z^2 \leq 1, x > 0$. H is symmetric with respect to reflection across the xz and yz planes, but not across the xy plane.

Let C be the solid cylinder $x^2 + y^2 \leq 1$ for z between 0 and 4. C is symmetric with respect to reflection across the xz and yz planes, but not across the xy plane. It is, however, symmetric with respect to reflection across the plane $z = 2$.

Let P be the part of the unit ball in the first octant. P is not symmetric with respect to any reflections across planes parallel to the coordinate planes.
We say a function g is **odd with respect to the reflection f** if $g(f(x, y, z)) = -g(x, y, z)$ for all points (x,y,z). We say a function g is **even with respect to the reflection f** if $g(f(x, y, z)) = g(x, y, z)$ for all points (x,y,z). Sometimes we just say g is **odd across a plane** if we mean that g is odd with respect to the reflection across the plane, and so on.

Here are some examples of functions with various odd and even symmetries:

Let $f = xyz^2$. When we replace x with $-x$, we get $f(-x, y, z) = (-xyz^2) = -(xyz^2) = -f(x, y, z)$, so f is odd with respect to reflection across the yz plane. When we replace z with $-z$, we get $f(x, y, -z) = (xy(-z)^2) = (xyz^2) = f(x, y, z)$, so f is even across the xy plane. Similarly, f is odd with respect to reflection across the xz plane.

Let $g = x^2 \sin(y) \ln(z)$. g is even with respect to reflection across the yz plane, odd with respect to reflection across the xz plane, and not symmetric with respect to reflection across the xy plane.
Let $h = \cos(z^2)xy$. h is even across the xy plane and odd across the xz and yz planes.

We can use symmetry in the following two ways:

1) If a region is symmetric with respect to a given reflection and the function we are integrating is odd with respect to the same reflection, the integral over the region will be zero! This can save us lots of work! It is zero because there will be cancellation from each side of the plane. One side will have positive integral and the other side will have a negative integral of exactly the same amount, so their sum will be zero.

2) If a region is symmetric with respect to a given reflection and the function we are integrating is even with respect to the same reflection, the integral over the region will be twice the integral of the function integrated over half of the region. The half of the region is the part that is on one side of the line or plane of reflection. This normally doesn't help too much as we still need to set up and do an integral.

So we can do integrals like:
$\iint_D x^2y^3\, dA = 0$ because f is odd with respect to reflection across the x-axis and the disc is symmetric across the x axis.

$\iiint_B x^2y\, dV = 0$ because f is odd with respect to reflection across the xz plane and the ball is symmetric across the xz plane.

$\iiint_H x^2 \sin(y) \ln(|z+1|)\, dV = 0$ because the function is odd with respect to reflection across the xz plane and the solid hemisphere H is symmetric across the xz plane.

$\iint_H x^2 \sin^2(y)|z|\, dV > 0$. For this one, we can't use symmetry to say that it is zero since the function is not odd with respect to any reflections. In fact, the function is always positive so the integral must be positive. So we will have to do some work to figure out what it is!

$\iiint_C x^2y^9e^z\, dV = 0$ because the function is odd with respect to reflection across the xz plane and the region is symmetric across the xz plane.

$\iiint_B (x^2 + y + z^3x^2 + \sin(y))\, dV = \iiint_B x^2\, dV$. For this one, we can break the integral up into 3 separate integrals. The first term is even with respect to the yz plane, so we can't do much with the first part. But the second term is odd with respect to the xz plane so its integral will be zero over the sphere. Also the third term is odd with respect to the xy plane so it will also have zero integral over the sphere. There is still some work to do, but we've made it easier!